

COHEN-MACAULAY, GORENSTEIN, COMPLETE INTERSECTION AND REGULAR DEFECT FOR THE TENSOR PRODUCT OF ALGEBRAS

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ABSTRACT. This paper main goal is to measure the defect of Cohen-Macaulayness, Gorensteiness, complete intersection and regularity for the tensor product of algebras over a ring. For this sake, we determine the homological invariants which are inherent to these notions, such as the Krull dimension, depth, injective dimension, type and embedding dimension of the tensor product constructions in terms of those of their components. Our results allow to generalize various theorems in this topic especially [BK1, Theorem 2.1], [TY, Theorem 6] and [M, Theorems 1 and 2] as well as two Grothendieck's theorems on the transfer of Cohen-Macaulayness and regularity to tensor products over a field issued from finite field extensions. To prove our theorems on the defect of complete intersection and regularity, the homology theory introduced by André and Quillen for commutative rings turns out to be an adequate and efficient tool in this respect.

1. INTRODUCTION

All rings of this paper are assumed to be unitary and Noetherian and all ring homomorphisms are unital. In particular, the tensor product $A \otimes_R B$ of two Noetherian algebras A and B over a Noetherian ring R is supposed to be Noetherian.

In [BK1], the authors were concerned with the problem of the transfer of the Cohen-Macaulayness property to the tensor product of algebras over a field. Its main theorem proves that the tensor product $A \otimes_k B$ of two algebras A and B over a field k , assuming Noetherianity of $A \otimes_k B$, is Cohen-Macaulay if and only if so are A and B . This theorem extends a result of Grothendieck in this regard when the components of the considered tensor product are two field extensions one of which is finitely generated over k . The transfer of the Gorenstein and complete intersection properties to these tensor product constructions was investigated by Watanabe, Ishikawa, Tachibana and Otsuka in [WITO] and subsequently by Tousi and Yassemi in [TY], proving similar theorems for these two notions as the above one on Cohen-Macaulayness.

The regularity property still resists to any effort of improvement at the level enjoyed by the above three notions. Indeed, contrary to the above notions, a Noetherian tensor product of two field extensions of a field k is not regular in general. In 1965, Grothendieck proved that $K \otimes_k L$ is a regular ring provided K or L is a finitely generated separable extension field of k [EGA IV, Seconde Partie, Lemma 6.7.4.1] and the finiteness hypotheses were dropped in [Sh1, Note]. In 1969, Watanabe,

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Ishikawa, Tachibana, and Otsuka, showed that under a suitable condition tensor products of regular rings are complete intersections [WITO, Theorem 2, p. 417] and subsequently Tousi and Yassemi proved that a Noetherian tensor product of two k -algebras A and B is regular if and only if so are A and B in the special case where k is perfect. In this context, the main theorem of [BK2] establishes necessary and sufficient conditions for a Noetherian tensor product of two extension fields of k to inherit regularity and hence generalizes Grothendieck's aforementioned result. Recently, a more general result (over a base ring) was obtained in [M].

In the stream of the above achievements, we aim in this paper to go farther in this direction by measuring the defect of Cohen-Macaulayness, Gorensteiness, complete intersection and regularity for the tensor product of algebras over a ring R . To this purpose, it is clearly essential to determine the very invariants which are inherent to these notions such as the Krull dimension, depth, injective dimension, type and embedding dimension of the tensor product constructions in terms of those of their components. That is our objective in the most part of this paper which allows us to evaluate the defect of the above notions for the tensor products under some flatness condition. Our results allow to generalize the above theorems in [BK1], [BK2], [M], [TY], [WITO]. In the last two sections of the paper we use the homology theory of André and Quillen. For the convenience of the reader, we include a whole section to give a brief introduction to such homology, including the results used in this paper.

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2. COHEN-MACAULAY RINGS

This section main goal is to determine the Cohen-Macaulayness defect of the tensor product $A \otimes_R B$ of algebras A and B over an arbitrary ring R in local and global cases. Also, we generalize [BK1, Theorem 2.1] which characterizes the Cohen-Macaulayness of the tensor product $A \otimes_k B$ of algebras A and B over a field k in terms of the Cohen-Macaulayness of A and B .

Let R be a ring and let A and B be R -algebras. First, it is worth noting that the tensor product $A \otimes_R B$ over R might be trivial even if A and B are not so. Of course, the interesting case is when $A \otimes_R B \neq \{0\}$ which makes it legitimate to seek conditions on the underlying components R , A and B of the tensor product which guarantee $A \otimes_R B \neq \{0\}$. That is the purpose of our first result.

For a given R -algebra A , denote by $f_A : R \rightarrow A$, with $f_A(r) = r \cdot 1_A$ for any $r \in R$ and where 1_A is the unit element of A , the ring homomorphism defining the structure of algebra of A over R . Also, if A and B are R -algebras, we denote by $\mu_A : A \rightarrow A \otimes_R B$ and $\mu_B : B \rightarrow A \otimes_R B$ the canonical algebra homomorphisms over A and B , respectively, such that $\mu_A(a) = a \otimes_R 1$ and $\mu_B(b) = 1 \otimes_R b$ for each $a \in A$ and each $b \in B$. Observe that the following diagram is commutative:

$$\begin{array}{ccccc}
 & & A & & \\
 & \nearrow f_A & & \searrow \mu_A & \\
 R & & f_{A \otimes_R B} & & A \otimes_R B \\
 & \searrow f_B & & \nearrow \mu_B & \\
 & & B & &
 \end{array}$$

Throughout this section, given ideals I, J and H of A, B and $A \otimes_R B$, respectively, we adopt the following notation for easiness: $I \cap R := f_A^{-1}(I)$, $J \cap R := f_B^{-1}(J)$ and $H \cap A := \mu_A^{-1}(H)$, $H \cap B := \mu_B^{-1}(H)$. By commutativity of the above diagram $(H \cap A) \cap R = (H \cap B) \cap R$. Also, given an ideal J of B , we denote by $A \otimes_R J$ the ideal $\mu_B(J)(A \otimes_R B)$ of $A \otimes_R B$ generated by $\mu_B(J) := \{1 \otimes_R b : b \in J\}$.

We begin by characterizing when the tensor product $A \otimes_R B$ of two algebras A and B over a ring R is not trivial. Also, we characterize when, given two prime ideals I and J of A and B , respectively, there exists a prime ideal P of $A \otimes_R B$ such that $P \cap A = I$ and $P \cap B = J$.

Proposition 2.1. *Let R be a ring and A, B be two R -algebras. Then*

(1) *$A \otimes_R B \neq \{0\}$ if and only if there exists a prime ideal I of A and a prime ideal J of B such that $I \cap R = J \cap R$.*

(2) *Let I be a prime ideal of A and J be a prime ideal of B . There exists a prime ideal P of $A \otimes_R B$ such that $P \cap A = I$ and $P \cap B = J$ if and only if $I \cap R = J \cap R$.*

Proof. (2) See [EGA I, Corollaire 3.2.7.1.(i)].

(1) Since $A \otimes_R B \neq \{0\}$ if and only if $\text{Spec}(A \otimes_R B) \neq \emptyset$, it follows from (2). \square

We denote by \mathcal{R} the class of Noetherian local rings (S, m, k) , where m is the maximal ideal of S and k its residue field. Let Λ consist of the functions

$$\lambda : \mathcal{R} \longrightarrow \mathbb{N} \cup \{+\infty\}$$

vanishing on any field and satisfying following property: for any flat local homomorphism $(S, m, k) \longrightarrow (T, n, l)$, we have $\lambda(T) = \lambda(S) + \lambda\left(\frac{T}{mT}\right)$. Since from the identity map of a field k we obtain $\lambda(k) = \lambda(k) + \lambda(k)$, the condition that λ vanishes on field k is equivalent to $\lambda(k) \neq +\infty$. The sum of two such functions is also an element of Λ as well as the difference (provided it is defined and takes values in $\mathbb{N} \cup \{+\infty\}$).

The following invariants are well known examples of elements of Λ .

Proposition 2.2. *The following functions from \mathcal{R} to $\mathbb{N} \cup \{+\infty\}$ are elements of Λ :*

- (1) *The Krull dimension (denoted by \dim).*
- (2) *The depth (denoted by depth).*
- (3) *The self-injective dimension (denoted by id).*
- (4) *The codepth (denoted by codepth).*
- (5) *The complete intersection defect (denoted by d).*

Proof. (1) See [Mt, Theorem 15.1].

(2) See [Mt, Corollary, p. 181] or [EGA IV, Seconde Partie, Proposition 6.3.1].

(3) See [FT, Corollary 1].

(4) Since $\text{codepth} = \dim - \text{depth}$, then (4) follows from (1) and (2).

(5) See [Av, Proposition 3.6]. \square

Let Λ' be the class of functions $\lambda' : \mathcal{R} \longrightarrow \mathbb{N} \cup \{+\infty\}$ verifying

$$\lambda'(R_n) + \lambda'((A \otimes_R B)_P) = \lambda'(A_p) + \lambda'(B_q) + \lambda'((k_A(p) \otimes_{k_R(n)} k_B(q))_{P(k_A(p) \otimes_{k_R(n)} k_B(q))})$$

and vanishing on any field (equivalently taking finite values on any field), where A and B are R -algebras and $P \in \text{Spec}(A \otimes_R B)$ with $p := P \cap A$, $q := P \cap B$ and $n := P \cap R$, and either A_p or B_q is flat over R_n .

We are going to see that $\Lambda = \Lambda'$.

Theorem 2.3. *Let A and B be two R -algebras. Let $P \in \text{Spec}(A \otimes_R B)$ with $p := P \cap A$, $q := P \cap B$ and $n := P \cap R$. Assume that A_p is flat over R_n . Then, for any $\lambda \in \Lambda$,*

$$\lambda((A \otimes_R B)_P) = \lambda\left(\frac{A_p}{nA_p}\right) + \lambda(B_q) + \lambda\left((k_A(p) \otimes_{k_R(n)} k_B(q))_{P(k_A(p) \otimes_{k_R(n)} k_B(q))}\right).$$

The proof of Theorem 2.3 requires the following preparatory lemma.

Lemma 2.4. *Let R be a ring. Let A and B be R -algebras.*

(1) *Let J be an ideal of B such that $J \cap R := m$ is a maximal ideal of R . Then*

$$\frac{A \otimes_R B}{A \otimes_R J} \cong \frac{A}{mA} \otimes_k \frac{B}{J} \cong A \otimes_R \frac{B}{J}$$

where k denotes the field $\frac{R}{m}$.

(2) *Let P be a prime ideal of $A \otimes_R B$. Let $p := P \cap A$, $q := P \cap B$ and $n := P \cap R$. Then*

$$\frac{(A \otimes_R B)_P}{(A \otimes_R q)_P} \cong \left(\frac{A_p}{nA_p} \otimes_{k_R(n)} k_B(q) \right)_{P\left(\frac{A_p}{nA_p} \otimes_{k_R(n)} k_B(q)\right)}.$$

Proof. (1) Since $A \otimes_R J = mA \otimes_R B + A \otimes_R J$ because $mA \otimes_R B \subset A \otimes_R J$, using the exact sequences

$$\begin{aligned} mA &\longrightarrow A \longrightarrow A/mA \longrightarrow 0 \\ J &\longrightarrow B \longrightarrow B/J \longrightarrow 0 \end{aligned}$$

it follows from [Bo, Chapter II, §3.6, Proposition 6] that

$$\frac{A \otimes_R B}{A \otimes_R J} \cong \frac{A \otimes_R B}{mA \otimes_R B + A \otimes_R J} \cong \frac{A}{mA} \otimes_R \frac{B}{J} \cong \left(\frac{A}{mA} \otimes_R k \right) \otimes_k \frac{B}{J} \cong \frac{A}{mA} \otimes_k \frac{B}{J}.$$

(2) First, note that $(A \otimes_R B)_P \cong (A_p \otimes_{R_n} B_q)_{P(A_p \otimes_{R_n} B_q)}$. Then

$$\frac{(A \otimes_R B)_P}{(A \otimes_R q)_P} \cong \frac{(A_p \otimes_{R_n} B_q)_{P(A_p \otimes_{R_n} B_q)}}{(A_p \otimes_{R_n} qB_q)_{P(A_p \otimes_{R_n} B_q)}} \cong \left(\frac{A_p \otimes_{R_n} B_q}{A_p \otimes_{R_n} qB_q} \right)_{\frac{P(A_p \otimes_{R_n} B_q)}{A_p \otimes_{R_n} qB_q}}.$$

Now, as $qB_q \cap R_n = nR_n$ is maximal in R_n , we get the desired isomorphism applying (1). \square

Proof of Theorem 2.3. First, observe that $(A \otimes_R B)_P \cong (A_p \otimes_{R_n} B_q)_{P(A_p \otimes_{R_n} B_q)}$. Then, without loss of generality, we may assume that R , A and B are local rings with maximal ideals n , p and q , respectively, such that A is flat over R , $P(A \otimes_R B)_P \cap A = p$, $P(A \otimes_R B)_P \cap B = q$ and $P(A \otimes_R B)_P \cap R = p \cap R = q \cap R = n$. As the ring homomorphism

$R \rightarrow A$ is flat, we get $B \rightarrow A \otimes_R B$ is a flat homomorphism which induces the flat homomorphism of rings $B \rightarrow (A \otimes_R B)_P$. Let $\lambda \in \Lambda$. Then, by hypothesis,

$$\begin{aligned} \lambda((A \otimes_R B)_P) &= \lambda(B) + \lambda\left(\frac{(A \otimes_R B)_P}{q(A \otimes_R B)_P}\right) \\ &= \lambda(B) + \lambda\left(\left(\frac{A \otimes_R B}{A \otimes_R q}\right)_{\frac{P}{A \otimes_R q}}\right). \end{aligned}$$

Hence, by Lemma 2.4, as $q \cap R = n$ is maximal in R , we get

$$\lambda((A \otimes_R B)_P) = \lambda(B) + \lambda\left(\left(\frac{A}{nA} \otimes_{k_R(n)} k_B(q)\right)_{P(\frac{A}{nA} \otimes_{k_R(n)} k_B(q))}\right).$$

Since

$$\frac{A}{nA} \longrightarrow \left(\frac{A}{nA} \otimes_{k_R(n)} k_B(q)\right)_{P(\frac{A_p}{nA_p} \otimes_{k_R(n)} k_B(q))}$$

is flat, we have

$$\lambda\left(\left(\frac{A}{nA} \otimes_{k_R(n)} k_B(q)\right)_{P(\frac{A}{nA} \otimes_{k_R(n)} k_B(q))}\right) = \lambda\left(\frac{A}{nA}\right) + \lambda\left(\left(k_A(p) \otimes_{k_R(n)} k_B(q)\right)_{P(k_A(p) \otimes_{k_R(n)} k_B(q))}\right).$$

It follows that

$$\lambda((A \otimes_R B)_P) = \lambda\left(\frac{A}{nA}\right) + \lambda(B) + \lambda\left(\left(k_A(p) \otimes_{k_R(n)} k_B(q)\right)_{P(k_A(p) \otimes_{k_R(n)} k_B(q))}\right)$$

completing the proof. \square

Corollary 2.5. $\Lambda = \Lambda'$.

Proof. If $\lambda \in \Lambda$, with the notation of Theorem 2.3, we have

$$\lambda\left(\frac{A_p}{nA_p}\right) + \lambda(R_n) = \lambda(A_p).$$

Then $\lambda \in \Lambda'$, by Theorem 2.3.

Now, if $\lambda' \in \Lambda'$, and $(S, m, k) \longrightarrow (T, n, l)$ is a flat local homomorphism, then

$$\lambda'(S) + \lambda'\left(\frac{T}{mT}\right) = \lambda'(S) + \lambda'(T \otimes_S k) = \lambda'(T) + \lambda'(k) + \lambda'(l \otimes_k k) = \lambda'(T)$$

since λ' vanishes on a field, so $\lambda' \in \Lambda$. \square

Our next result deals with the Krull dimension as well as the depth of local tensor products of algebras over an arbitrary ring R . The case of a base field was obtained in [BK1, Proposition 2.3].

Corollary 2.6. *Let A and B be R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that A_p is flat over R_n . Then*

$$\begin{aligned} (1) \dim((A \otimes_R B)_P) &= \dim(A_p) + \dim(B_q) - \dim(R_n) + \\ &\quad \dim\left(\left(k_A(p) \otimes_{k_R(n)} k_B(q)\right)_{P(k_A(p) \otimes_{k_R(n)} k_B(q))}\right). \\ (2) \text{depth}((A \otimes_R B)_P) &= \text{depth}(A_p) + \text{depth}(B_q) - \text{depth}(R_n) + \\ &\quad \dim\left(\left(k_A(p) \otimes_{k_R(n)} k_B(q)\right)_{P(k_A(p) \otimes_{k_R(n)} k_B(q))}\right). \end{aligned}$$

Proof. By Proposition 2.2, $\dim \in \Lambda$ and $\text{depth} \in \Lambda$. Then the result follows from Corollary 2.5 since the Krull dimension and the depth are finite invariants when applied to Noetherian local rings, in other terms, $\dim, \text{depth}: \mathcal{R} \longrightarrow \mathbb{N}$. \square

Remark 2.7. The Krull dimension of $k_A(p) \otimes_{k_R(n)} k_B(q)$ is the minimum of the transcendence degrees of the field extensions $k_A(p)|k_R(n)$ and $k_B(q)|k_R(n)$. This was proved in [EGA IV, Quatrième Partie, Err_{VI}.19, page 349]. See also [Sh2] and [Mu].

Recall that the “codepth” of a Noetherian local ring A denoted by $\text{codepth}(A) := \dim(A) - \text{depth}(A)$ is an invariant introduced by Grothendieck in [EGA IV, Première Partie, 0_{IV}.16.4.9] to measure the Cohen-Macaulayness defect of A . Next, we determine the codepth of local tensor products of algebras over a ring.

Corollary 2.8. *Let A and B be R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that A_p is flat over R_n . Then*

$$\begin{aligned} \text{codepth}((A \otimes_R B)_P) &= \text{codepth}(A_p) + \text{codepth}(B_q) - \text{codepth}(R_n) \\ &= \text{codepth}\left(\frac{A_p}{nA_p}\right) + \text{codepth}(B_q). \end{aligned}$$

Proof. The first equality follows from Corollary 2.6 and the second from Proposition 2.2. \square

We end this section by discussing the Cohen-Macaulayness of tensor products of algebras over a ring R . We begin with the local case and characterize when the localization of $A \otimes_R B$ is Cohen-Macaulay.

Corollary 2.9. *Let A and B be R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that A_p is flat over R_n . Then the following assertions are equivalent:*

- (1) $(A \otimes_R B)_P$ is a Cohen-Macaulay ring;
- (2) $\frac{A_p}{nA_p}$ and B_q are Cohen-Macaulay rings;
- (3) B_q is a Cohen-Macaulay ring and $\text{codepth}(A_p) = \text{codepth}(R_n)$.

Proof. It is direct by the preceding corollary. \square

Corollary 2.10. *Let A and B be R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that A_p and B_q are flat over R_n . Then the following assertions are equivalent:*

- (1) $(A \otimes_R B)_P$ is a Cohen-Macaulay ring;
- (2) A_p and B_q are Cohen-Macaulay rings.

Proof. (2) \Rightarrow (1) It is direct by Corollary 2.9 as $R_n \rightarrow A_p$ is a flat homomorphism. (1) \Rightarrow (2) By the above corollary B_q is Cohen-Macaulay, and then so is A_p by symmetry. \square

Corollary 2.11. *Let A and B be R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that A_p is flat over R_n and R_n is a Cohen-Macaulay ring. Then the following assertions are equivalent:*

- (1) $(A \otimes_R B)_P$ is a Cohen-Macaulay ring;
- (2) A_p and B_q are Cohen-Macaulay rings.

Next, we deal with the global case of Cohen-Macaulayness of tensor products. The following corollaries generalize [BK1, Theorem 2.1] which characterizes the Cohen-Macaulayness of $A \otimes_k B$ in terms of the Cohen-Macaulayness of A and B in the setting of a field k .

Corollary 2.12. *Let A and B be R -algebras. Assume that A is flat over R . Then the following assertions are equivalent:*

- (1) $A \otimes_R B$ is a Cohen-Macaulay ring;
- (2) $\frac{A_p}{nA_p}$ and B_q are Cohen-Macaulay rings for any prime ideals p of A and q of B such that $p \cap R = q \cap R = n$.

Proof. (1) \Rightarrow (2) Assume that (1) holds. Let $p \in \text{Spec}(A)$ and $q \in \text{Spec}(B)$ such that $p \cap R = q \cap R =: n$. Then by Proposition 2.1 there exists a prime ideal P of $A \otimes_R B$ such that $P \cap A = p$ and $P \cap B = q$. By (1), $A \otimes_R B$ is Cohen-Macaulay, thus $(A \otimes_R B)_P$ is Cohen-Macaulay. As a consequence, by Corollary 2.9, B_q and $\frac{A_p}{nA_p}$ are Cohen-Macaulay.

(2) \Rightarrow (1) Assume that (2) holds. Let P be a prime ideal of $A \otimes_R B$ and let $n := P \cap R$, $p := P \cap A$, $q := P \cap B$. As $p \cap R = q \cap R = P \cap R = n$, we get by (2) that $\frac{A_p}{nA_p}$ and B_q are Cohen-Macaulay rings. Hence, by Corollary 2.9, $(A \otimes_R B)_P$ is Cohen-Macaulay. It follows that $A \otimes_R B$ is Cohen-Macaulay completing the proof. \square

Corollary 2.13. *Let A and B be R -algebras. Assume that A and B are flat over R . Then the following assertions are equivalent:*

- (1) $A \otimes_R B$ is a Cohen-Macaulay ring;
- (2) A_p and B_q are Cohen-Macaulay rings for any prime ideals p of A and q of B such that $p \cap R = q \cap R$.

Proof. It is similar to that of Corollary 2.12 using Corollary 2.10. \square

Corollary 2.14. *Let R be a Cohen-Macaulay ring. Let A and B be R -algebras. Assume that A is flat over R . Then the following assertions are equivalent:*

- (1) $A \otimes_R B$ is a Cohen-Macaulay ring;
- (2) A_p and B_q are Cohen-Macaulay rings for any prime ideals p of A and q of B such that $p \cap R = q \cap R$.

Proof. Again, the proof is similar to that of Corollary 2.12, but using Corollary 2.11. \square

3. GORENSTEIN RINGS

The aim of this section is to determine the injective dimension of local tensor products of algebras over an arbitrary ring R . This permits to generalize [TY, Theorem 6] on Gorensteiness of tensor products of algebras over a field.

We begin by announcing the main theorem of this section. It gives the injective dimension of local tensor products of algebras over a ring R .

Theorem 3.1. *Let A and B be R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that A_p is flat over R_n . Then*

$$\text{id}_{(A \otimes_R B)_P}((A \otimes_R B)_P) = \text{id}_{\frac{A_p}{nA_p}}\left(\frac{A_p}{nA_p}\right) + \text{id}_{B_q}(B_q) + \dim\left((k_A(p) \otimes_{k_R(n)} k_B(q))_{P(k_A(p) \otimes_{k_R(n)} k_B(q))}\right).$$

Proof. The proof is straightforward from Theorem 2.3 since, by Proposition 2.2, $\text{id} \in \Lambda$, and using Bass's formula [B, Lemma 3.3] we have that

$$\text{id}_D(D) = \text{depth}(D) = \dim(D),$$

as $D := (k_A(p) \otimes_k k_B(q))_{P(k_A(p) \otimes_k k_B(q))}$ is a Gorenstein local ring by [TY, Theorem 6]. \square

The formula for the injective dimension of local tensor products of algebras over a field k is simpler as recorded next.

Corollary 3.2. *Let k be a field. Let A and B be k -algebras. Let P be a prime ideal of $A \otimes_k B$ with $p := P \cap A$ and $q := P \cap B$. Then*

$$\mathrm{id}_{(A \otimes_k B)_P}((A \otimes_k B)_P) = \mathrm{id}_{A_p}(A_p) + \mathrm{id}_{B_q}(B_q) + \dim\left((k_A(p) \otimes_k k_B(q))_{P(k_A(p) \otimes_k k_B(q))}\right).$$

The remaining of this section is devoted to discuss the Gorensteiness of tensor products of algebras over a ring R in the local case as well as in the global one. In particular, we generalize [TY, Theorem 6] which proves that a tensor product $A \otimes_k B$ of algebras A and B over a field k is Gorenstein if and only if A and B are so. The following corollaries are direct consequences of Theorem 3.1 so that the proofs are omitted. It suffices to recall that $(k_A(p) \otimes_{k_R(n)} k_B(q))_{P(k_A(p) \otimes_{k_R(n)} k_B(q))}$ is a Gorenstein local ring.

Corollary 3.3. *Let A and B be R -algebras. Let $P \in \mathrm{Spec}(A \otimes_R B)$ and let $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that $R_n \rightarrow A_p$ is a flat ring homomorphism. Then the following assertions are equivalent:*

- (1) $(A \otimes_R B)_P$ is a Gorenstein ring;
- (2) $\frac{A_p}{nA_p}$ and B_q are Gorenstein rings.

Corollary 3.4. *Let A and B be R -algebras. Let $P \in \mathrm{Spec}(A \otimes_R B)$ and let $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that $R_n \rightarrow A_p$ is a flat ring homomorphism and that R_n is a Gorenstein ring. Then the following assertions are equivalent:*

- (1) $(A \otimes_R B)_P$ is a Gorenstein ring;
- (2) A_p and B_q are Gorenstein rings.

Corollary 3.5. *Let A and B be R -algebras. Assume that $R \rightarrow A$ is a flat ring homomorphism. Then the following assertions are equivalent:*

- (1) $A \otimes_R B$ is a Gorenstein ring;
- (2) $\frac{A_p}{nA_p}$ and B_q are Gorenstein rings for any prime ideals p of A and q of B such that $p \cap R = q \cap R$.

Corollary 3.6. *Let R be a Gorenstein ring. Let A and B be R -algebras. Assume that A and B are flat over R . Then the following assertions are equivalent:*

- (1) $A \otimes_R B$ is a Gorenstein ring;
- (2) A_p and B_q are Gorenstein rings for any prime ideals p of A and q of B such that $p \cap R = q \cap R$.

4. TYPE OF LOCAL RINGS

Let (R, m, k) be a local ring, where m is the maximal ideal of R and k its residue field. Recall that the type of an R -module M , denoted by $r_R(M)$, is an invariant which refines the information given by the depth of M , namely $r_R(M) := \dim_k(\mathrm{Ext}_R^t(k, M))$ where $t = \mathrm{depth}_R(M)$. In this section, we express the type of the tensor product of algebras over a ring in terms of the type of its components.

We begin by recording the following theorem which examines the behavior of the type under flat ring homomorphisms.

Theorem 4.1. *Let $(R, m, k) \longrightarrow (S, n, l)$ be a flat ring homomorphism of local Noetherian rings. Then*

$$r_S(S) = r_R(R) \cdot r_{\frac{S}{mS}}\left(\frac{S}{mS}\right).$$

Proof. See [FT, Theorem]. □

The following result is a direct consequence of this theorem.

Corollary 4.2. *The composed function \ln or of the the Neperian logarithm \ln and the type r is an element of Λ .*

Recall that, by [BH, Theorem 3.2.10], a local ring A is Gorenstein if and only if A is a Cohen-Macaulay ring of type 1¹. From this perspective, the type of local rings might be regarded as an invariant which measures the defect of Gorensteiness of local rings. The following result examines the behavior of the property of being “of type 1” under flat ring homomorphisms.

Corollary 4.3. *Let $(R, m, k) \longrightarrow (S, n, l)$ be a flat ring homomorphism of local Noetherian rings. Then S is of type 1 if and only if so are R and $\frac{S}{mS}$.*

Proof. The proof is straightforward from Theorem 4.1. □

Next, we announce the main theorem of this section. It computes the type of a local tensor product of algebras over a ring R in terms of the types of their components. Given a ring A and when no confusion is likely, we denote by $r(A)$ the type $r_A(A)$ of A over itself.

Theorem 4.4. *Let A and B be R -algebras. Let P be a prime ideal of $A \otimes_R B$ and let $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that A_p is flat over R_n . Then*

$$r((A \otimes_R B)_P) = r\left(\frac{A_p}{nA_p}\right) \cdot r(B_q) = \frac{r(A_p) \cdot r(B_q)}{r(R_n)}.$$

Proof. The proof follows from Theorem 4.1, Theorem 2.3 and Corollary 4.2, noting that $(k_A(p) \otimes_{k_R(n)} k_B(q))_{Pk_A(p) \otimes_{k_R(n)} k_B(q)}$ is a Gorenstein local ring and thus of type 1. □

Now, we record when a local tensor product of algebras over a ring behaves nicely with respect to Gorensteiness, namely when it is of type 1.

Corollary 4.5. *Let A and B be R -algebras. Let P be a prime ideal of $A \otimes_R B$ and let $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that A_p is flat over R_n . Then $(A \otimes_R B)_P$ is of type 1 if and only if $\frac{A_p}{nA_p}$ and B_q are of type 1.*

Proof. It is clear from Theorem 4.4. □

Our last results of this section determine the type of local tensor products of algebras over a field k and examine when these constructions are of type 1.

¹ Note that if the type is defined as $r_R(R) := \dim_k(\text{Ext}_R^t(k, R))$ where $t = \dim R$ instead of $t = \text{depth}_R(R)$, then a local ring is Gorenstein if and only if it is of type 1 [R].

Corollary 4.6. *Let k be a field and let A and B be k -algebras. Let P be a prime ideal of $A \otimes_k B$. Let $p := P \cap A$ and $q := P \cap B$. Then*

$$r((A \otimes_k B)_P) = r(A_p) \cdot r(B_q).$$

Corollary 4.7. *Let k be a field and let A and B be k -algebras. Let P be a prime ideal of $A \otimes_k B$. Let $p := P \cap A$ and $q := P \cap B$. Then $(A \otimes_k B)_P$ is of type 1 if and only if A_p and B_q are of type 1.*

5. INTRODUCTION TO ANDRÉ-QUILLEN HOMOLOGY

This section represents a short introduction to André-Quillen homology. If $A \rightarrow B$ is a ring homomorphism and M is a B -module, then $H_n(A, B, M)$ denotes the André-Quillen homology B -module for each $n \geq 0$ [An]. In the following, we list the basic properties of these homology modules used in the remaining sections.

Properties. (i) (Functoriality) *They are natural in all three variables A , B and M .*

(ii) *If $A \rightarrow B$ is an isomorphism, then $H_n(A, B, M) = 0$ for all $n \geq 0$. If $B \cong \frac{A}{I}$, then $H_0(A, B, M) = 0$ and*

$$H_1(A, B, M) \cong \frac{I}{I^2} \otimes_B M.$$

In particular, if (A, m, k) is a Noetherian local ring, $\dim_F H_1(A, k, F) = \text{embdim}(A)$ for any field extension $F|k$.

(iii) (Base change) *Let $A \rightarrow B$, $A \rightarrow C$ be ring homomorphisms and t an integer such that $\text{Tor}_n^A(B, C) = 0$ for all $0 < n < t$. Then for any $B \otimes_A C$ -module M , the canonical homomorphisms*

$$H_n(A, B, M) \rightarrow H_n(C, B \otimes_A C, M)$$

are isomorphisms for all $n < t$.

(iv) *Let $A \rightarrow B \rightarrow C$ be ring homomorphisms and M a flat C -module. Then*

$$H_n(A, B, M) \cong H_n(A, B, C) \otimes_C M$$

for all n .

(v) (Jacobi-Zariski exact sequence) *Let $A \rightarrow B \rightarrow C$ be ring homomorphisms and M a C -module. Then there exist a natural exact sequence*

$$\begin{aligned} \cdots &\rightarrow H_n(A, B, M) \rightarrow H_n(A, C, M) \rightarrow H_n(B, C, M) \rightarrow \\ &\rightarrow H_{n-1}(A, B, M) \rightarrow \cdots \\ \cdots &\rightarrow H_0(A, B, M) \rightarrow H_0(A, C, M) \rightarrow H_0(B, C, M) \rightarrow 0 \end{aligned}$$

(vi) (Tensor product) *Let B and C be A -algebras and t an integer such that $\text{Tor}_n^A(B, C) = 0$ for all $0 < n < t$. Then for any $B \otimes_A C$ -algebra D and any D -module M we have a natural exact sequence*

$$\begin{aligned} &H_t(B \otimes_A C, D, M) \rightarrow \\ \rightarrow &H_{t-1}(A, D, M) \rightarrow H_{t-1}(B, D, M) \oplus H_{t-1}(C, D, M) \rightarrow H_{t-1}(B \otimes_A C, D, M) \rightarrow \cdots \\ &\cdots \rightarrow H_0(B \otimes_A C, D, M) \rightarrow 0 \end{aligned}$$

(vii) (Field extensions) *If $L|K$ is a field extension and M an L -module, then $H_n(K, L, M) = 0$ for all $n \geq 2$. Moreover, $L|K$ is separable if and only if $H_1(K, L, M) = 0$ for some (equivalently any) L -module $M \neq 0$. Therefore if $A \rightarrow K \rightarrow L$ are ring homomorphisms with K and L fields, then by (v) $H_n(A, K, M) \rightarrow H_n(A, L, M)$ is an isomorphism for all $n \geq 2$ and*

injective for $n = 1$. If $L \mid K$ is separable, it is also an isomorphism for $n = 1$.

(viii) (Localization) Let $f: A \longrightarrow B$ be a ring homomorphism and let $S \subset A$, $T \subset B$ be multiplicatively closed subsets such that $f(S) \subset T$. Let M be a B -module, then

$$\begin{aligned} T^{-1}H_n(A, B, M) &\cong H_n(A, B, T^{-1}M) \cong H_n(A, T^{-1}B, T^{-1}M) \\ &\cong H_n(S^{-1}A, T^{-1}B, T^{-1}M) \end{aligned}$$

for all $n \geq 0$.

(ix) (Finiteness) If B an A -algebra of finite type and M a B -module of finite type, then $H_n(A, B, M)$ is a B -module of finite type for all $n \geq 0$.

(x) (Vanishing) A Noetherian local ring (A, m, k) is regular if and only if $H_2(A, k, k) = 0$ if and only if $H_n(A, k, k) = 0$ for all $n \geq 2$. If A is a ring and (B, n, l) a local A -algebra, then B is formally smooth for the n -adic topology [EGA IV, Première Partie, 0_{IV}.19.3.1] if and only if $H_1(A, B, l) = 0$.

(xi) (Flat extensions) If $(A, m, k) \longrightarrow (B, n, l)$ is a local flat homomorphism of local rings, then the homomorphism $H_2(A, k, l) \longrightarrow H_2(B, l, l)$ is injective.

(xii) (Completions) Let (A, m, k) and (B, n, l) be two local rings with their completions \hat{A} and \hat{B} . Then

$$H_n(A, k, k) \cong H_n(\hat{A}, k, k) \quad \text{and} \quad H_n(A, B, l) \cong H_n(\hat{A}, \hat{B}, l)$$

for all n .

Proof. (i) See [An, 3.15].

(ii) See [An, 4.43, 4.60, 6.1].

(iii) See [An, 9.31].

(iv) See [An, 3.20].

(v) See [An, 5.1].

(vi) Same proof than in [An, 5.21], but using [An, 9.31] instead of [An, 4.54].

(vii) See [An, 7.4, 7.13].

(viii) The first isomorphism is [An, 4.59] and the last two follow from [An, 5.27].

(ix) See [An, 4.55].

(x) [An, 6.26] gives the first claim. The second follows from [An, 16.17, 3.20, 3.21].

(xi) See [Av, Remarks (1.4)].

(xii) The first isomorphism is [An, 10.18] and the second one is an immediate consequence of the first (a reference for this isomorphism is [FR, Lemma 1]). \square

6. COMPLETE INTERSECTION

The aim of this section is to evaluate the complete intersection defect of the tensor product of two algebras over a ring.

Let (S, m, l) be a local ring. The *complete intersection defect* of S [KK] is defined as

$$d(S) := \varepsilon_2(S) - \text{embdim}(S) + \dim(S),$$

where $\varepsilon_2(S)$ denotes $\dim_l H_2(S, l, l)$, which is finite by Property (ix). It is remarkable that, by [An, 15.12] (see also [MR, 2.5.1]), $\varepsilon_2(S)$ coincides with the dimension of the first Koszul homology module, denoted by $\dim H_1(m)$, associated to a minimal set of generators of the ideal m . Also, notice that $d(S) \geq 0$, and $d(S) = 0$ if and only if S is complete intersection [KK, p. 349] (or in terms of André-Quillen homology see [MR, 4.3.4, 4.3.5]).

Recall that, in [M, Theorem 2], it is proved the following theorem on complete intersection of tensor products: Let A and B two R -algebras such that for each maximal

ideal P of $A \otimes_R B$, with $n = P \cap R$, $p = P \cap A$ and $q = P \cap B$, A_p or B_q is flat over R_n . If A and B are complete intersection then so is $A \otimes_R B$. Our next theorem generalizes this result by evaluating the complete intersection defect of such constructions.

Theorem 6.1. *Let A and B be R -algebras. Let $P \in \text{Spec}(A \otimes_R B)$ with $p := P \cap A$, $q := P \cap B$ and $n := P \cap R$. Assume that A_p is flat over R_n . Then*

$$d((A \otimes_R B)_P) = d(A_p) + d(B_q) - d(R_n).$$

Proof. It is immediate from Proposition 2.2 and Corollary 2.5 as

$(k_A(p) \otimes_{k_R(n)} k_B(q))_{P(k_A(p) \otimes_{k_R(n)} k_B(q))}$ is complete intersection. \square

Next, we isolate the case of tensor products of algebras over a field.

Corollary 6.2. *Let k be a field and let A and B be k -algebras. Let $P \in \text{Spec}(A \otimes_k B)$ with $p := P \cap A$ and $q := P \cap B$. Then*

$$d((A \otimes_k B)_P) = d(A_p) + d(B_q).$$

The next corollaries characterizes when the tensor product of algebras over a ring is complete intersection.

Corollary 6.3. *Let A and B be R -algebras. Let $P \in \text{Spec}(A \otimes_R B)$ with $p := P \cap A$, $q := P \cap B$ and $n := P \cap R$. Assume that A_p is flat over R_n . Then the following assertions are equivalent:*

- (1) $(A \otimes_R B)_P$ is complete intersection;
- (2) $\frac{A_p}{nA_p}$ and B_q are complete intersections;
- (3) B_q is complete intersection and $d(A_p) = d(R_n)$.

Proof. It suffices to note, by Theorem 6.1, that $d((A \otimes_R B)_P) = d\left(\frac{A_p}{nA_p}\right) + d(B_q)$. \square

Corollary 6.4. *Let A and B be R -algebras. Let $P \in \text{Spec}(A \otimes_R B)$ with $p := P \cap A$, $q := P \cap B$ and $n := P \cap R$. Assume that A_p and B_q are flat over R_n . Then the following assertions are equivalent:*

- (1) $(A \otimes_R B)_P$ is complete intersection;
- (2) A_p and B_q are complete intersections.

Proof. It is direct as if either A_p or B_q is complete intersection, then so is R_n . \square

Note that one may characterize when the tensor product $A \otimes_R B$ is complete intersection by using Proposition 2.1 as done in Section 2 for Cohen-Macaulayness.

Let $(R, n, k_R(n))$ be a Noetherian local ring and \hat{R} its completion. It is well known (see [K]) that for any representation of \hat{R} as $\hat{R} = \frac{S}{I}$, where S is a regular local ring and I an ideal of R , we have

$$d(R) = \mu(I) - \dim(S) + \dim(R).$$

Here, $\mu(I) = \dim H_1(S, \hat{R}, k_R(n))$ stands for the minimum number of generators of I (Property (ii)). R is said to be an *almost complete intersection* ring if $d(R) = 1$.

The following two corollaries characterize when the tensor product of two R -algebras is almost complete intersection.

Corollary 6.5. *Let A and B be R -algebras. Let $P \in \text{Spec}(A \otimes_R B)$ with $p := P \cap A$, $q := P \cap B$ and $n := P \cap R$. Assume that A_p is flat over R_n . Then $(A \otimes_R B)_P$ is almost complete intersection if and only if one of the following assertions holds:*

- (1) $\frac{A_p}{nA_p}$ is complete intersection and B_q is almost complete intersection.
- (2) $\frac{A_p}{nA_p}$ is almost complete intersection and B_q is complete intersection.

Proof. It is clear as by Theorem 6.1, $d((A \otimes_R B)_P) = d\left(\frac{A_p}{nA_p}\right) + d(B_q)$. □

Note that under the conditions of Corollary 6.5 if R_n , A_p and B_q are almost complete intersection, then so is $(A \otimes_R B)_P$.

Corollary 6.6. *Let A and B be R -algebras. Let $P \in \text{Spec}(A \otimes_R B)$ with $p := P \cap A$, $q := P \cap B$ and $n := P \cap R$. Assume that A_p and B_q are flat over R_n . Then $(A \otimes_R B)_P$ is almost complete intersection if and only if one of the following assertions holds:*

- (1) R_n , A_p and B_q are almost complete intersection rings.
- (2) R_n and A_p are complete intersection and B_q is almost complete intersection.
- (3) R_n and B_q are complete intersection and A_p is almost complete intersection.

Proof. First, as by Theorem 6.1, $d((A \otimes_R B)_P) = d\left(\frac{A_p}{nA_p}\right) + d(B_q) = d(A_p) + d\left(\frac{B_q}{nB_q}\right)$, note that if $(A \otimes_R B)_P$ is almost complete intersection, then $d(R_n) \leq 1$, $d(A_p) \leq 1$ and $d(B_q) \leq 1$. If R_n is complete intersection, then, by Corollary 6.5, $(A \otimes_R B)_P$ is almost complete intersection if and only if either A_p is almost complete intersection and B_q is complete intersection or A_p is complete intersection and B_q is almost complete intersection, as desired. Now, assume that R_n is almost complete intersection. As $d(A_p) = d(R_n) + d\left(\frac{A_p}{nA_p}\right) \geq 1$, if $d(A_p) \leq 1$, then $d(A_p) = 1$, that is, A_p is almost complete intersection. Similarly, if $d(B_q) \leq 1$, then B_q is almost complete intersection. It follows that $(A \otimes_R B)_P$ is almost complete intersection if and only if A_p and B_q are so. This completes the proof of the corollary. □

7. REGULAR RINGS

We are mainly concerned in this section with measuring the defect of regularity of the tensor product of algebras over a ring.

For a local ring S , we denote by $\text{codim } S := \text{embdim } S - \dim S$ the codimension of S , so that S is regular if and only if $\text{codim } S = 0$. Thereby, from the very definition of complete intersection defect of S , note that

$$\varepsilon_2(S) = d(S) + \text{codim}(S)$$

where $\varepsilon_2(S) = \dim_l H_2(S, l, l)$. Therefore $\varepsilon_2(S)$, as well as $\text{codim}(S)$, is a measure of the defect of regularity of S (it also follows from Property (x)).

Lemma 7.1. *Let A and B be two R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that A_p is a flat R_n -module. Then, the following statements are equivalent:*

- (1) $\varepsilon_2((A \otimes_R B)_P) = \varepsilon_2(A_p) + \varepsilon_2(B_q) - \varepsilon_2(R_n)$;
- (2) $\text{codim}((A \otimes_R B)_P) = \text{codim}(A_p) + \text{codim}(B_q) - \text{codim}(R_n)$.

Proof. It suffices to note that, by Theorem 6.1,

$$d((A \otimes_R B)_P) = d(A_P) + d(B_q) - d(R_n)$$

and thus

$$\begin{aligned} \varepsilon_2((A \otimes_R B)_P) - \varepsilon_2(A_P) - \varepsilon_2(B_q) + \varepsilon_2(R_n) &= \text{codim}((A \otimes_R B)_P) - \text{codim}(A_P) \\ &\quad - \text{codim}(B_q) + \text{codim}(R_n). \end{aligned}$$

□

Recently, in [M, Theorem 1], it is proved the following theorem on regularity of tensor products: *Let R be a Noetherian ring and let A and B be two Noetherian R -algebras such that $A \otimes_R B$ is a Noetherian ring. Assume that for each maximal ideal P of $A \otimes_R B$ with $n = P \cap R$, $q = P \cap A$ and $q = P \cap B$ at least one of the three local R_n -algebras A_P , B_q or $k_{A \otimes_R B}(P)$ is formally smooth for the topology of its maximal ideal. If A and B are regular rings, then so is $A \otimes_R B$.* Our next results generalize this theorem by measuring the defect of regularity of these tensor products under the formal smoothness of one the components over the base ring.

We begin by dealing with the codimension of rings issued from formally smooth homomorphisms.

Theorem 7.2. *Let A and B be two R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. If $R_n \rightarrow A_p$ is formally smooth, then*

$$\text{codim}((A \otimes_R B)_P) = \text{codim}(A_P) + \text{codim}(B_q) - \text{codim}(R_n).$$

Proof. Assume that $R_n \rightarrow A_p$ is formally smooth. In light of Lemma 7.1, it suffices to prove that $\varepsilon_2((A \otimes_R B)_P) = \varepsilon_2(A_P) + \varepsilon_2(B_q) - \varepsilon_2(R_n)$. Observe that, by [EGA IV, Première Partie, 0_{IV}.19.7.1], A_p is a flat R_n -module. Now, using Property (vi), consider the exact sequence (where E is the residue field of $(A \otimes_R B)_P$)

$$\begin{aligned} H_2(R_n, E, E) &\xrightarrow{\alpha_2} H_2(A_p, E, E) \oplus H_2(B_q, E, E) \rightarrow H_2(A_p \otimes_{R_n} B_q, E, E) \rightarrow \\ &\rightarrow H_1(R_n, E, E) \xrightarrow{\alpha_1} H_1(A_p, E, E) \oplus H_1(B_q, E, E). \end{aligned}$$

Let us show that α_1 and α_2 are injective. Note that α_2 is injective by Property (xi). Also, Property (v) yields the following exact sequence

$$H_1(R_n, A_p, E) = 0 \xrightarrow{\beta} H_1(R_n, E, E) \rightarrow H_1(A_p, E, E)$$

by Property (x) as $R_n \rightarrow A_p$ is formally smooth. Therefore α_1 is injective. Hence the following sequence is exact

$$0 \rightarrow H_2(R_n, E, E) \rightarrow H_2(A_p, E, E) \oplus H_2(B_q, E, E) \rightarrow H_2(A_p \otimes_{R_n} B_q, E, E) \rightarrow 0.$$

As $H_2(A_p \otimes_{R_n} B_q, E, E) \cong H_2((A \otimes_R B)_P, E, E)$ by Property (viii), and $\dim_E H_2(A_p, E, E) = \varepsilon_2(A_P)$ by Property (vii) (and similarly for R_n and B_q), it follows that

$$\varepsilon_2((A \otimes_R B)_P) = \varepsilon_2(A_P) + \varepsilon_2(B_q) - \varepsilon_2(R_n)$$

completing the proof of the theorem. □

Corollary 7.3. *Let $(A, m, k) \rightarrow (B, n, l)$ be a formally smooth homomorphism of local rings. Then*

$$(1) \quad \varepsilon_2(B) = \varepsilon_2(A) + \varepsilon_2\left(\frac{B}{mB}\right).$$

$$(2) \quad \text{codim}(B) = \text{codim}(A) + \text{codim}\left(\frac{B}{mB}\right).$$

Proof. Part (2) follows from Theorem 7.2 and Lemma 7.1, since

$$\text{codim}\left(\frac{B}{mB}\right) = \text{codim}(B \otimes_A k) = \text{codim } B - \text{codim } A + \text{codim } k = \text{codim } B - \text{codim } A$$

and (1) follows from (2) as above. \square

The following corollaries discuss the regularity property as well as the embedding dimension of the tensor products of algebras over a ring under formal smoothness hypothesis.

Corollary 7.4. *Let A and B be two R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. Assume that $R_n \rightarrow A_p$ is formally smooth. Consider the following assertions:*

- (1) A_p and B_q are regular.
 - (2) $(A \otimes_R B)_P$ is regular;
 - (3) $\frac{A_p}{nA_p}$ and B_q are regular;
 - (4) B_q is regular and $\text{codim}(A_p) = \text{codim}(R_n)$.
- Then (1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4).

Proof. It is direct from Theorem 7.2 as by Corollary 7.3, $\text{codim}(A_p) = \text{codim}(R_n) + \text{codim}\left(\frac{A_p}{nA_p}\right)$. \square

Corollary 7.5. *Let A and B be two R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $n := P \cap R$, $p := P \cap A$ and $q := P \cap B$. If $R_n \rightarrow A_p$ is formally smooth, then*

$$\begin{aligned} \text{embdim}((A \otimes_R B)_P) &= \text{embdim}(A_p) + \text{embdim}(B_q) - \text{embdim}(R_n) \\ &\quad + \dim\left((k_A(p) \otimes_{k_R(n)} k_B(q))_{P(k_A(p) \otimes_{k_R(n)} k_B(q))}\right). \end{aligned}$$

Proof. It follows from Corollary 2.6 and Theorem 7.2. \square

Next, we announce our second main theorem of this section.

Theorem 7.6. *Let A and B be R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $p := P \cap A$ and $q := P \cap B$. Assume that $R_n \rightarrow k_A(p)$ is formally smooth, that is, R_n is a field and $k_A(p)$ is separable over R_n . Then*

- (1) $\text{codim}((A \otimes_R B)_P) = \text{codim}(A_p) + \text{codim}(B_q)$.
- (2) $\text{embdim}((A \otimes_R B)_P) = \text{embdim}(A_p) + \text{embdim}(B_q) + \dim\left((k_A(p) \otimes_{R_n} k_B(q))_{P(k_A(p) \otimes_{R_n} k_B(q))}\right)$.

Proof. The proof of (1) is the same as that of Theorem 7.2, taking in mind that the homomorphism β in that proof factorizes as

$$H_1(R_n, A_p, E) \rightarrow H_1(R_n, k_A(p), E) \rightarrow H_1(R_n, E, E)$$

and $H_1(R_n, k_A(p), E) = 0$ by Property (vii). Then (2) follows from (1) and Corollary 2.6. \square

Corollary 7.7. *Let A and B be R -algebras. Let P be a prime ideal of $A \otimes_R B$ with $p := P \cap A$ and $q := P \cap B$. Assume that $R_n \rightarrow k_A(p)$ is formally smooth (i.e., R_n is a field and $k_A(p)$ is separable over R_n). Then the following assertions are equivalent:*

- (1) $(A \otimes_R B)_P$ is regular;
- (2) A_p and B_q are regular.

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